orphisms of surfaces of algebraically finite type by Morse-Smale diffeomorphisms with

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In this paper, we describe the realization of each homotopy class of type T_2 by a Morse-Smale diffeomorphism with an orientable heteroclinic set. Such diffeomorphisms have relatively simple dynamics, since, by virtue of the results of A.N. Bezdezhnykh and V.Z. Grines, have only a finite number of heteroclinic orbits. Moreover, we prove that the type of the homotopy class of any Morse-Smale diffeomorphism with a finite number of heteroclinic orbits is uniquely determined by the index of its heteroclinic intersection.

Let $S_{g,k}$, $g \geq 0$, $k \geq 0$ – be a connected compact orientable surface of genus g with the boundary consisting of k connected components. We set $S_{g,0} = S_g$. Everywhere below, surface mappings are assumed to preserve orientation.

A homeomorphism $h : S_{g,k} \to S_{g,k}$ is called a *periodic homeomorphism* if there exists $m \in \mathbb{N}$, such that $h^m = id$, where id — is the identity transformation. The smallest of these numbers m is called the *period* of the periodic homeomorphism.

A homeomorphism $h : S_g \to S_g$, $g \geq 1$ is called a *reducible* by system C of disjoint simple closed curves C_i, $i = 1, \ldots, l$, non-homotopic to zero and pairwise not homotopic to each other if the system of curves C is invariant under h.

A reducible nonperiodic homeomorphism $h : S_g \to S_g$, $g \geq 1$ is called a *homeomorphism of algebraically finite type*, f there exists an h-invariant neighborhood $\mathbb{C}$ of curves of the set C, which consists of the union two-dimensional anulus and such that for each connected component S_{g_j,k_j}, $j = 1, \ldots, n$ of the set $S_g \setminus int\, \mathbb{C}$ there is a number $m_j \in \mathbb{N}$ such that $h^{m_j}|_{S_{g_j,k_j}} : S_{g_j,k_j} \to S_{g_j,k_j}$ – is a periodic homeomorphism.

Recall that a diffeomorphism $f : S_g \to S_g$ is called a *Morse-Smale diffeomorphism* if

1) the non-wandering set Ω_f consists of a finite number of hyperbolic orbits;

2) the invariant manifolds W^s_p, W^u_q intersect transversally for any non-wandering points p, q.

Denote by $MS(S_g)$ the set of Morse-Smale diffeomorphisms. In the set of periodic orbits of any diffeomorphism $f \in MS(S_g)$ one can introduce a total order relation, which is a continuation of the partial order introduced by S. Smale [6]. Precisely, let $\mathcal{O}_i, \mathcal{O}_j$ — be the periodic orbits of the Morse-Smale diffeomorphism f. They say that the orbits $\mathcal{O}_i, \mathcal{O}_j$ *are in the relation* $\prec$ ($\mathcal{O}_i \prec \mathcal{O}_j$), if

$$W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset.$$

A sequence of different periodic orbits $\mathcal{O}_i = \mathcal{O}_{i_0}, \mathcal{O}_{i_1}, ..., \mathcal{O}_{i_k} = \mathcal{O}_j$ $(k \geqslant 1)$, such that $\mathcal{O}_{i_0} \prec \mathcal{O}_{i_1} \prec \ldots \prec \mathcal{O}_{i_k}$ is called a *chain of length* k*, connecting periodic orbits* $\mathcal{O}_i$ *and* $\mathcal{O}_j$. The chain connecting the periodic orbits of saddle points will be called *saddle chain*. Since the non-wandering set is finite, for any diffeomorphism $f \in MS(M^n)$ there is a well-defined number equal to the length of the maximal saddle chain, which is denoted by

$$beh(f).$$

Let σ_i, σ_j — be saddle points of the diffeomorphism f such that $W^s_{\sigma_i} \cap W^u_{\sigma_j} \neq \emptyset$. Recall that the intersection $W^s_{\sigma_i} \cap W^u_{\sigma_j}$ is a countable set and each point of this set is called *heteroclinic point*, and each orbit of a heteroclinic point is called a *heteroclinic orbit*. For any heteroclinic point $x \in W^s_{\sigma_i} \cap W^u_{\sigma_j}$ For any heteroclinic point $(\vec{v}^u_x, \vec{v}^s_x)$, where:

- $\vec{v}^u_x$ — the tangent vector to the unstable manifold of the point σ_j at the point x;

- $\vec{v}^s_x$ — the tangent vector to the stable manifold of the point σ_i at the point x.

Following [1](or see for example [2, p. 7]), we call a heteroclinic intersection of the diffeomorphism f *orientable*, if the ordered pairs of vectors $(\vec{v}^u_x, \vec{v}^s_x)$ set the same orientation of the bearing surface S_g. Otherwise, the heteroclinic intersection is called *non-orientable*.

Two homeomorphisms $h, h' : S_g \to S_g$ are called *homotopic*, if there exists a continuous mapping $H : S_g \times [0, 1] \to S_g$ such that $H(x, 0) = h(x)$? $H(x, 1) = h'(x)$. By $[h]$ we denote the *homotopy class* of the homeomorphism h.

Theorem 1. *In every homotopy class* $[h]$ *of the homeomorphism* $h : S_g \to S_g$, $g \geq 1$ *of algebraically finite type, there exists a Morse-Smale diffeomorphism* $f : S_g \to S_g$ *with orientable heteroclinic intersection.*

In [4], it was announced and then proved in [3] that any diffeomorphism $f \in MS(S_g)$ with orientable heteroclinic intersections has $beh(f) = 1$. This fact was also proved in the work [5] using the factorization method.

Let $f : S_g \to S_g$ be an orientation-preserving Morse-Smale diffeomorphism such that $beh(f) \leq 1$ (that is, the diffeomorphism f has a finite number of heteroclinic orbits). Let us denote by $MS_1(S_g)$ the set of such diffeomorphisms. By virtue of [7], the dynamics of any diffeomorphism $f \in MS_1(S_g)$ can be represented as follows.

The set Ω_f of periodic orbits of the maps f can be divided into subsets $\Omega^i_f, i \in \{\omega, s, u, \alpha\}$ as follows:

* Ω^ω_f — the set of all sink orbits;

* Ω^s_f — s the set of saddle orbits whose unstable manifolds do not contain heteroclinic points;

* Ω^u_f — the set of the remaining saddle orbits of the system;

* Ω^α_f — the set of source orbits.

Let

$$\mathcal{A}_f = \Omega^\omega_f \cup W^u_{\Omega^s_f},\ \mathcal{R}_f = \Omega^\alpha_f \cup W^s_{\Omega^u_f},\ V_f = S_g \setminus (\mathcal{A}_f \cup \mathcal{R}_f).$$

By construction, all heteroclinic points of the diffeomorphism f belong to the set V_f, which consists of a finite number of connected components V_i, $i = 1, \ldots, m$. Each component V_i is homeomorphic to an open two-dimensional ring and is invariant with respect to some power $q_i \in \mathbb{N}$ of the diffeomorphism f. Each heteroclinic orbit $\mathcal{O}_x \subset V_i$ of the diffeomorphism f^{q_i} is assigned the index $\xi_{\mathcal{O}_x}$, equal to $+1(-1)$, if the orientation of the carrier the surface (not) coincides with the orientation defined by the pair of vectors $(\vec{v}^u_x, \vec{v}^s_x)$. Since the diffeomorphism f preserves orientation, the index $\xi_{\mathcal{O}_x}$ does not depend on the choice of a point in the orbit $\mathcal{O}_x$. We set

$$\xi_i = \sum_{\mathcal{O}_x \subset V_i} \xi_{\mathcal{O}_x},\ \xi_f = \sum_{i=1}^{m} |\xi_i|$$

and we will call the number ξ_f *the index of the heteroclinic intersection* of the diffeomorphism $f \in MS_1(S_g)$.

It follows directly from the definition that the heteroclinic intersection index is a non-negative number. The next result shows that it uniquely determines the type of the homotopy class $[f]$ of the diffeomorphism $f \in MS_1(S_g)$.

Theorem 2. *Let $f \in MS_1(S_g)$. Then $[f]$ is of type T_1, if $\xi_f = 0$ and $[f]$ is of type T_2, if $\xi_f > 0$.*

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